MATH6031 Lecture 8

<u>SYZ</u> : Assuming $\exists$ a (special) Lagrangian torus fibration $\begin{array}{c} X \\ \downarrow \\ B \end{array}$ w/o singular fibers,

then mirror symmetry works as a T-duality:

$$\begin{tikzcd}[column sep=small] (X \cong T^*B/\Lambda^\vee, \omega_0) \arrow[dr] & & (TB/\Lambda =: \check{X}, \check{J}_0) \arrow[dl] \\ & B & \end{tikzcd}$$

semi-flat SYZ mirror

<u>Rmk</u> (Hitchin) Define the McLean metric by

$$g(v_1, v_2) := -\int_{L_b} \iota_{v_1}\omega \wedge \iota_{v_2} \mathrm{Im}\,\Omega \quad \text{Hessian metric}$$

$$\overset{\text{(locally)}}{=} \frac{\partial^2 \phi}{\partial x_i \partial x_j}, \quad \phi : \text{convex fcn on } B$$

- observation : the sympl. and complex affine structures are related by the Legendre transform:

$$\xi_j = \frac{\partial \phi}{\partial x_j}, \quad j = 1, \dots, n$$

- A Hessian metric on B ϕ determines a complex structure J_0 on X (which makes (X, ω_0, J_0) Kähler) and a sympl. structure $\check{\omega}_0$ on $\check{X}$ (which makes $(\check{X}, \check{\omega}_0, \check{J}_0)$ Kähler).

- Also, the T^n-invariant metric on X (and $\check{X}$) is Ricci-flat $\iff$ $\det\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) = \text{const.}$ real Monge-Ampère eqn.

(cf. works of Cheng-Yau, Greene-Shapere-Vafa-Yau).

SYZ transform

Now we have dual torus fibrations

$$\begin{tikzcd} T^*B/\Lambda^\vee = X \arrow[dr, "\pi"'] & & \check{X} = TB/\Lambda \arrow[dl, "\check{\pi}"] \\ & B & \end{tikzcd}$$

Key pt : $(\check{\pi})^{-1}(b) = (\pi^{-1}(b))^\vee = \left\{ \begin{array}{l} \text{flat } U(1)\text{-conn.} \\ \text{on } \mathbb{C} \to \pi^{-1}(b) \end{array} \right\} \Big/ \begin{array}{l}\text{gauge} \\ \text{equiv.}\end{array}$

$$\vec{y} = (y_1, \ldots, y_n) \longmapsto \nabla_{\vec{y}} = d + \frac{i}{2} \sum_{j=1}^n y_j \, du_j$$

$\rightsquigarrow$ Poincaré line bundle / universal $U(1)$-conn. on $X \times_B \check{X}$

$$d + \frac{i}{2} \sum_{j=1}^n (y_j \, du_j - u_j \, dy_j)$$

whose curvature is given by

$$F = i \sum_{j=1}^n dy_j \wedge du_j$$

Consider the fiber product

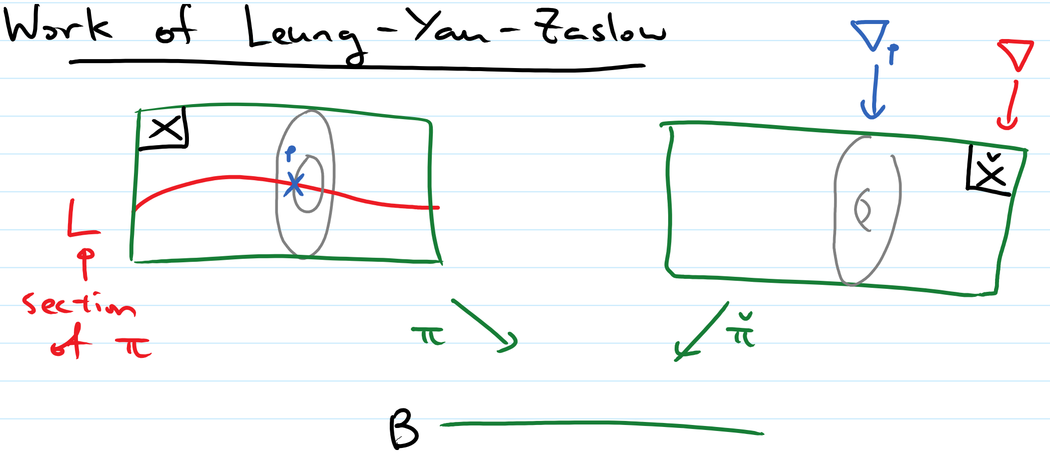
$$\begin{tikzcd} & X \times_B \check{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "\pi"'] & & \check{X} \arrow[dl, "\check{\pi}"] \\ & B & \end{tikzcd}$$

and define the semi-flat SYZ transform :

$$\mathcal{F}_{SYZ} : \Omega^\cdot(X) \longrightarrow \Omega^\cdot(\check{X})$$
$$\alpha \longmapsto (p_2)_* \left(p_1^*(\alpha) \wedge e^{iF} \right)$$

Exercise $\quad \mathcal{F}_{SYZ}(e^{i\omega_0}) = \check{\Omega}_0 := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$

Work of Leung-Yau-Zaslow



a section $L \subset X$ of π $\longleftrightarrow$ a $U(1)$-conn ∇ over $\check{X}$

Exercise (LYZ): L is Lagr. $\iff$ $F^{0,2} = (\nabla^2)^{0,2} = 0$

So mirror symmetry gives a corr.

$$\left\{ \begin{matrix} \text{Lagr sections} \\ L \subset X \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{holom. line} \\ \text{bundles} / \check{X} \end{matrix} \right\}$$

$\longrightarrow$ HMS

Further references: Leung (2005)
for MS w/o corrections Abouzaid (201x)

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§ SYZ w/ corrections I

Setting (D. Auroux 2007)

(X, D) where X: Kähler $\qquad K_X^{-1} = \Lambda^{top} T_X$

$D \in |K_X^{-1}|$

$D \subset X$: effective anticanonical divisor (simple normal crossing)

e.g. • $X = (\mathbb{C}^\times)^n$, $D = 0$.

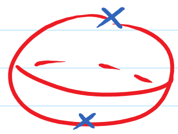
• $X = \mathbb{C}^n$, $D = \{z_1 \cdots z_n = 0\}$

toric prime divisors

- $X = X_\Delta$ toric Kähler, $D = D_{toric} = \bigcup_{i=1} D_i$

  e.g. $X = \mathbb{P}^n$, $D = \bigcup_{i=0}^{n} \{z_i = 0\}$

  (say $X = \mathbb{P}^1$, $D = \{2 \text{ pts}\}$)



Assumptions

① $\exists$ a Lagrangian torus fibration

$$\begin{array}{ccccc} D & \subset & X & \supset & X \setminus D \\ \downarrow & & \downarrow & & \downarrow \\ \partial B & \subset & B & \supset & B_0 = B \setminus \partial B \end{array}$$

w/o singular fibers over the interior $B_0 := B \setminus \partial B$.

(so that B is an affine manifold with boundary).

② Maslov class $\mu(L_b) = 0$ for any $b \in B_0$.

$\uparrow$

$\arg(\Omega|_L) : L_b \to S^1$

$\exists\, \Omega$: nowhere vanishing meromorphic volume form w/ only simple poles along D.

$(\Rightarrow X \setminus D$ is CY$)$

Rmk : This is a necessary condition for a Lagr. to be special.

③ L_b does not bound any noncontant holom. disks in $X \setminus D$.

(assumption on the Floer theory of L_b).

Rmk : All the above examples satisfy these assumptions.

Now we study mirror symmetry for (X, D).

First of all,

$$\begin{tikzcd} X_0 := X \setminus D \arrow[d] \\ B_0 := B \setminus \partial B \end{tikzcd}$$

is a Lagr. torus fibration w/o singular fibers

$\longrightarrow$ semi-flat mirror symmetry

$$\begin{tikzcd} X \supset X_0 = T^*B_0/\Lambda^\vee \arrow[dr] & & \check{X}_0 := TB_0/\Lambda \arrow[dl] \\ & B_0 & \end{tikzcd}$$

semi-flat mirror of $X_0 = X \setminus D$.

$\check{X}_0$ captures only part of the mirror because we lose information of D!

Indeed, in $X \setminus D = X_0$, the Lagr torus fibers are tautologically unobstructed (as objects in $\mathrm{Fuk}(X \setminus D)$)

However, they may become obstructed in $\mathrm{Fuk}(X)$ since they may bound holom. disks in X.

This suggests that the remaining part of the mirror is captured by the Floer theory of torus fibers as objects in $\mathrm{Fuk}(X)$.

§ Fukaya-Oh-Ohta-Ono (FOOO)'s formulation of Floer theory

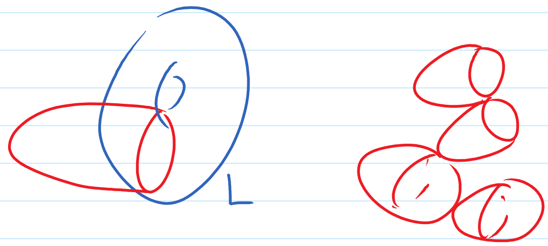
Let $L \subset X \setminus D$ be a Lagrangian torus fiber, and ∇ be a flat $U(1)$-conn on L.

Consider

$$\mathcal{M}_k(L,\beta) = \text{moduli space of holom. disks in } X \text{ with boundary on } L \text{ and class } \beta \in \pi_2(X,L)$$

$$= \{ u : (D, \partial D) \to (X, L) \mid [u] = \beta \} / \sim$$
$$p_1, \ldots, p_k \in \partial D$$

& $\overline{\mathcal{M}}_k(L,\beta)$ = compactification of $\mathcal{M}_k(L,\beta)$ by adding bubbled configurations



$$\text{vir.}\dim_{\mathbb{R}} \overline{\mathcal{M}}_k(L,\beta) = n + \mu(\beta) + k - 3$$

where $\mu(\beta) = 2 \cdot D \cdot \beta \geq 0$ is the Maslov index of β.

We define the Floer complex of (L, ∇)

$$\left(C^\bullet((L,\nabla),\Lambda), \{m_k\}_{k \geq 0} \right) \text{ as follows :}$$

distribution valued k-forms on L

- $C^k((L,\nabla),\Lambda) = \bar{S}^k((L,\nabla),\Lambda) = \mathrm{Im}\left(S_{n-k}(L;\mathbb{Q}) \to D^k(L;\mathbb{R}) \right) \otimes_{\mathbb{Q}} \Lambda$

  $(n-k)$-dim$^{\text{nl}}$ chains on L

$$[P, f] \longmapsto T([P,f])$$
$$\omega \longmapsto \int_P f^* \omega$$
$$\Omega^{n-k}(L)$$

- $m_k : \underbrace{C^\bullet(L,\nabla) \times \cdots \times C^\bullet(L,\nabla)}_{k} \to C^\bullet(L,\nabla)$

  is defined as follows

  - $m_{k,\beta}([P_1,f_1],\ldots,[P_k,f_k]) := \pm \left[\overline{\mathcal{M}}_{k+1}(L,\beta) {}_{(ev_1,\ldots,ev_k)}\times_{(f_1,\ldots,f_k)} (P_1 \times \cdots \times P_k), ev_0 \right]$

  - Define

$$m_k([P_1,f_1],\ldots,[P_k,f_k]) := \sum_{\substack{\beta \in \pi_2(X,L) \\ \beta \neq 0}} m_{k,\beta}([P_1,f_1],\ldots,[P_k,f_k]) \cdot Z_\beta$$

where $Z_\beta(L,\nabla) := e^{-\int_\beta \omega} \mathrm{hol}_\nabla(\partial\beta)$

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$\leftarrow$ holom. fcn on $\check{X}_0 = \{(L,\nabla)\}/\sim$

for $k \neq 1, 2$

$$m_1([P,f]) = d([P,f]) + \sum_{\substack{\beta \in \pi_2(X,L) \\ \beta \neq 0}} m_{1,\beta}([P,f])\, Z_\beta$$

$$m_2([P_1,f_1],[P_2,f_2]) = [P_1,f_1] \wedge [P_2,f_2] + \sum_{\substack{\beta \in \pi_2(X,L) \\ \beta \neq 0}} m_{2,\beta} \cdot Z_\beta$$

- $\{m_k\}_{k \geq 0}$ satisfy the curved A_∞-relations :

$$m_1(m_0) = 0$$

$$\boxed{m_1^2(-) = \cancel{m_2(-, \overset{c\cdot[L]}{m_0}) \pm m_2(\overset{c\cdot[L]}{m_0}, -)}}$$

$$\begin{aligned} &m_1(m_2(-,-)) \pm m_2(m_1(-),-) \pm m_2(-,m_1(-)) \\ &= \pm m_3(m_0,-,-) \pm m_3(-,m_0,-) \pm m_3(-,-,m_0) \end{aligned}$$

$$\vdots$$

- m_0 is called the obstruction chain :

$$m_0 = m_0(L,\nabla) = \sum_{\substack{\beta \in \pi_2(X,L) \\ \beta \neq 0}} (ev_0)_*\left([\overline{\mathcal{M}}_1(L,\beta)]^{vir}\right) \cdot Z_\beta(L,\nabla)$$

Def (L,∇) is called weakly unobstructed (resp. unobstructed)

if $m_0 = $ const. multiple of $[L]$

(resp. $m_0 = 0$)

If (L,∇) is weakly unobstructed, then $m_1^2 = 0$

$\rightsquigarrow$ $HF^\bullet((L,\nabla)) := H^\bullet(C^\bullet((L,\nabla)), m_1)$ is well-defined

Floer cohomology.

Actually, $(C^\bullet(L,\nabla), \{m_k\}_{k\geq 1})$ is a genuine A_∞-algebra

Actually, $(C(L,\nabla), \{m_k\}_{k\geq 1})$ is a genuine A_∞-algebra

Prop (Seidel, FOOO, Auroux)
In the above situation, (L,∇) is weakly unobstructed.

Pf: By assumption, $\mu(\beta) = 2\cdot(\beta\cdot D) \geq 2$ if $\beta \neq 0$.

$\Longrightarrow$ $[\overline{\mathcal{M}}_1(L,\beta)]^{vir}$ is a cycle if $\mu(\beta)=2$

and $\mathrm{vir}\dim_{\mathbb{R}} \overline{\mathcal{M}}_1(L,\beta) = n + \mu(\beta) + 1 - 3$

$= n + \mu(\beta) - 2 > n$ if $\mu(\beta) > 2$

$$\Longrightarrow m_0(L,\nabla) = \sum_{\mu(\beta)=2} \underbrace{(ev_0)_*\Big(\underbrace{[\overline{\mathcal{M}}_1(L,\beta)]^{vir}}_{\dim = n \;\Downarrow}\Big)}_{n_\beta\cdot[L]} \cdot Z_\beta(L,\nabla)$$

is a multiple of $[L]$. #

Write $m_0(L,\nabla) = W(L,\nabla)\cdot[L]$.

Then we call

$$W(L,\nabla) = \sum_{\substack{\beta\in\pi_2(X,L)\\ \mu(\beta)=2}} n_\beta \cdot Z_\beta(L,\nabla)$$

the Lagrangian Floer superpotential.

Assuming convergence (and working over $\mathbb{C}$ instead of Λ),

the superpotential $W(L,\nabla)$ is a holom. fcn

on $\check{X}_0 = TB_0/\Lambda = \left\{(L,\nabla) \;\middle|\; \begin{array}{l} L : \text{fiber of } \begin{smallmatrix} X_0 \\ \downarrow \\ B_0 \end{smallmatrix} \\ \nabla : \text{flat } U(1)\text{-conn on } L \end{array}\right\}\Big/\sim$

The Landau-Ginzburg model $(\check{X}_0, W)$

is the SYZ mirror of (X,D).

e.g. mirror of $\left(\mathbb{P}^2, D = \bigcup_{i=0}^{2} \{z_i = 0\}\right)$

is given by $\left((\mathbb{C}^\times)^2, W = z_1 + z_2 + \frac{q}{z_1 z_2}\right)$.